4.5a Routh-Hurwitz criteria

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Def. (Hurwitz matrix)

Given the polynomial $P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$, $a_i \in \mathbb{R}$, the nth Hurwitz matrix of the polynomial is

$a_0 = 1$
$a_i = 0$ for $i < 0$
$a_i = 0$ for $i > n$

$$H_n = \begin{bmatrix} a_1 & a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+2} \\ a_3 & a_2 & a_1 & a_0 & \cdots & a_{-n+4} \\ a_5 & a_4 & a_3 & a_2 & \cdots & a_{-n+6} \\ \vdots & & & \ddots & & \vdots \\ a_{2n-1} & a_{2n-2} & \cdots & & \cdots & a_n \end{bmatrix}$$

And the n Hurwitz matrices for $k < n$ are given by the principal minors of H_n. i.e.

$$H_1 = [a_1], \quad H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} a_1 & a_0 & a_{-1} \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}$$

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & \cdots & & 0 \\ a_3 & a_2 & a_1 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

Thm 4.4 (Routh-Hurwitz criteria)

All the roots of the polynomial $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ have negative real parts iff $\det(H_j) > 0$ for $j = 1, 2, \ldots, n$.

proof. Beyond scope of course.

proof for $n=2$: $P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0$

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{\ldots} \qquad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{\ldots}$$

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \qquad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}.$$

Routh-Hurwitz criteria is $\det(H_1) = \det(a_1) > 0 \implies \boxed{a_1 > 0}$

$$\det(H_2) = \det\begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix} = \det\begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} = a_1 a_2 > 0$$

$$\implies \boxed{a_2 > 0.}$$

Forward case: Let $a_1 > 0$, $a_2 > 0$.

Suppose $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_1 = \bar{\lambda}$. Then $Re(\lambda_1) = Re(\lambda_2) = \frac{-a_1}{2} < 0$.

Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $a_1^2 - 4a_2 \geq 0$.

Clearly, $\lambda_2 < 0$ since both $-a_1 < 0$ and $-\sqrt{a_1^2 - 4a_2} \leq 0$.

Also, $a_1^2 - 4a_2 < a_1^2$ since $a_2 > 0$.

$$\implies \sqrt{a_1^2 - 4a_2} < a_1 \implies -a_1 + \sqrt{a_1^2 - 4a_2} < 0$$

$$\implies \lambda_1 < 0.$$

Backward case: Let $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$.

Suppose $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_1 = \bar{\lambda}_2$. Then $Re(\lambda_1) = Re(\lambda_2) = \frac{-a_1}{2} \implies a_1 > 0$.

Also $a_1^2 - 4a_2 < 0 \implies 4a_2 > a_1^2 > 0 \implies a_2 > 0$.

Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $\lambda_1, \lambda_2 < 0$ and $a_1^2 - 4a_2 \geq 0$.

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} < 0 \implies \frac{-a_1}{2} < 0 \implies a_1 > 0.$$

$$\implies -a_1 + \sqrt{a_1^2 - 4a_2} < 0$$

$$\implies \sqrt{a_1^2 - 4a_2} < a_1$$

$$\implies a_1^2 - 4a_2 < a_1^2$$

$$\Rightarrow -4a_2 < 0 \Rightarrow a_2 > 0. \quad \boxed{}$$

Ex. $n=2$, $a_1>0$, $a_2>2$

$n=3$, $a_1>0$, $a_3>0$, $a_1a_2>a_3$

$n=4$, $a_1>0$, $a_3>0$, $a_4>0$, $a_1a_2a_3 > a_3^2 + a_1^2a_4$

$n=5$,
$$\begin{cases} a_i>0, & i=1,2,3,4,5 \\ a_1a_2a_3 > a_3^2 + a_1^2a_4 \\ (a_1a_4-a_5)(a_1a_2a_3-a_3^2-a_1^2a_4) > a_5(a_1a_2-a_3)^2 + a_1a_5^2. \end{cases}$$

Corollary 4.1 Given $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, $a_i \in \mathbb{R}$, if all of the roots of $P(\lambda)$ have negative real parts, then $a_i>0$ for $i=1,2,\ldots,n$.

Corollary of Thm 4.4.

Direct proof. Let $-r_1, -r_2, \ldots, -r_{K_1}$ be the real roots of $P(\lambda)$, counting multiplicity. Let $-c_1 \pm id_1, \ldots, -c_{K_2} \pm id_{K_2}$ be the complex roots, counting multiplicity. $c_j, d_j, r_j \in \mathbb{R}$. Note that $r_j, c_j > 0$ by assumption.

Then
$$P(\lambda) = (\lambda+r_1)\cdots(\lambda+r_{K_1})\left[(\lambda+c_1-id_1)(\lambda+c_1+id_1)\right]\cdots\left[(\lambda+c_{K_2}-id_{K_2})(\lambda+c_{K_2}+id_{K_2})\right]$$
$$= (\lambda+r_1)\cdots(\lambda+r_{K_1})(\lambda^2+2c_1\lambda+c_1^2+d_1^2)\cdots(\lambda^2+2c_{K_2}\lambda+c_{K_2}^2+d_{K_2}^2).$$

Note that all coefficients in factored equation are positive (>0), so after multiplying out, $a_1, \ldots, a_n > 0$. $\boxed{}$